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RIGIDITY OF COMPOSITION OPERATORS ON THE HARDY SPACE H^p

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Abstract. Let ϕ be an analytic map taking the unit disk $\mathbb{D}$ into itself. We establish that the class of composition operators $f \mapsto C_\phi(f) = f \circ \phi$ exhibits a rather strong rigidity of non-compact behaviour on the Hardy space H^p, for $1 \leq p < \infty$ and $p \neq 2$. Our main result is the following trichotomy, which states that exactly one of the following alternatives holds: (i) C_ϕ is a compact operator $H^p \to H^p$, (ii) C_ϕ fixes a (linearly isomorphic) copy of ℓ^p in H^p, but C_ϕ does not fix any copies of ℓ^2 in H^p, (iii) C_ϕ fixes a copy of ℓ^2 in H^p. Moreover, in case (iii) the operator C_ϕ actually fixes a copy of $L^p(0,1)$ in H^p provided $p > 1$. We reinterpret these results in terms of norm-closed ideals of the bounded linear operators on H^p, which contain the compact operators $\mathcal{K}(H^p)$. In particular, the class of composition operators on H^p does not reflect the quite complicated lattice structure of such ideals.

1. Introduction and preliminaries

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in $\mathbb{C}$. For $0 < p < \infty$ the analytic function $f : \mathbb{D} \to \mathbb{C}$ belongs to the Hardy space H^p if

$$\|f\|_p^p = \sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(r\xi)|^p dm(\xi) < \infty, \tag{1.1}$$

where $\mathbb{T} = \partial\mathbb{D}$ (identified with $[0, 2\pi]$) and $dm(e^{it}) = \frac{dt}{2\pi}$. Let $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map of $\mathbb{D}$. It is a well-known consequence of the Littlewood subordination principle, see e.g. [9, 3.1], that the composition operator

$$f \mapsto C_\phi(f) = f \circ \phi$$

is bounded $H^p \to H^p$ for any ϕ as above. Properties of these composition operators have been studied very extensively during the last 40 years on various Banach spaces of analytic functions on $\mathbb{D}$, see [9] and [34] for comprehensive expositions of the early developments of the area. The compactness of C_ϕ on H^p is well understood, and there are several equivalent characterisations in the literature. To exhibit a specific criterion recall that Shapiro [33] established

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that C_ϕ is a compact operator $H^p \to H^p$ if and only if

$$\lim_{|w|\to 1} \frac{N(\phi, w)}{\log(1/|w|)} = 0. \tag{1.2}$$

Above $N(\phi, w)$ is the Nevanlinna counting function of ϕ defined by $N(\phi, w) = \sum_{z\in\phi^{-1}(w)} \log(1/|z|)$ for $w \in \phi(\mathbb{D})$ (counting multiplicities). Finer gradations of compactness were obtained e.g. by Luecking and Zhu [26], who characterised the membership of C_ϕ in the Schatten p-classes on H^2. Moreover, the approximation numbers of C_ϕ on H^2 were estimated in [19], [20] and [21], as well as on H^p in [22].

The purpose of this paper is to demonstrate that composition operators on H^p only allow a small variety of qualitative non-compact behaviour compared to that of arbitrary bounded operators on H^p. Let E, F and X be Banach spaces. It will be convenient to say that the bounded linear operator $U : E \to F$ *fixes a copy of* X in E if there is an infinite-dimensional subspace $M \subset E$, M linearly isomorphic to X, for which $U_{|M}$ is bounded below on M, that is, there is $c > 0$ so that $\|Ux\| \geq c \cdot \|x\|$ for all $x \in M$. We use the standard notation $M \approx X$ for linearly isomorphic spaces M and X, and refer to [1], [23] and [39] for general background related to the theory of Banach spaces.

The trichotomy contained in Theorem 1.1 below is the main result of this paper. Let $E_\phi = \{e^{i\theta} : |\phi(e^{i\theta})| = 1\}$ be the boundary contact set of the analytic map $\phi : \mathbb{D} \to \mathbb{D}$. Here, and in the sequel, we use $\phi(e^{i\theta})$ to denote the a.e. radial limit function of ϕ on $\mathbb{T}$. It is part of the trichotomy that (1.2) together with the simple condition

$$m(E_\phi) = 0 \tag{1.3}$$

completely determine the composition operators which fix copies of the subspace ℓ^p or ℓ^2 in H^p. Recall that the known compactness results for C_ϕ on H^2 yield that (1.2) implies (1.3), but the class of symbols ϕ satisfying (1.3) is much larger than that of (1.2), see e.g. [34, Chap. 10].

In the statement below we exclude the Hilbert space H^2, where the situation is known and much simpler, since part (ii) does not occur for $p = 2$ (cf. the discussion following Theorem 1.2). We use $\mathcal{K}(E)$ to denote the class of compact operators $E \to E$ for any Banach space E, and take into account the known characterisation of the composition operators $C_\phi \in \mathcal{K}(H^p)$.

Theorem 1.1. *Let $1 \leq p < \infty$, $p \neq 2$, and ϕ be any analytic self-map of $\mathbb{D}$. Then there are three mutually exclusive alternatives:*

(i) *C_ϕ is compact on H^p,*
(ii) *C_ϕ fixes a copy of ℓ^p in H^p, but does not fix any copies of ℓ^2 in H^p,*
(iii) *C_ϕ fixes a copy of ℓ^2 (as well as of ℓ^p) in H^p. In this case, if $1 < p < \infty$ and $p \neq 2$, then C_ϕ also fixes a copy of $L^p(0, 1)$ in H^p.*

Furthermore, regarding the above alternatives

(i) *takes place if and only if Shapiro's condition* (1.2) *holds,*
(ii) *takes place if and only if* (1.2) *fails to hold but* $m(E_\phi) = 0$,
(iii) *takes place if and only if* $m(E_\phi) > 0$.

In particular, $C_\phi \in \mathcal{K}(H^p)$ *if and only if* C_ϕ *does not fix any copies of* ℓ^p *in* H^p.

Theorem 1.1 is obtained by combining Theorems 1.2, 1.4 and 1.5 stated below, which also contain more precise information. For this purpose we first recall some standard linear classes that classify the behaviour of non-compact operators. Let E, F and X be Banach spaces, and $\mathcal{L}(E, F)$ be the space of bounded linear operators from E to F. The operator $U \in \mathcal{L}(E, F)$ is called X-*singular* if U does not fix any copies of X in E. We denote

$$\mathcal{S}_X(E, F) = \{U \in \mathcal{L}(E, F) : U \text{ is } X\text{-singular}\},$$

and put $\mathcal{S}_p(E, F) = \mathcal{S}_{\ell^p}(E, F)$ to simplify our notation in the case of $X = \ell^p$. Recall further that $U \in \mathcal{L}(E, F)$ is *strictly singular*, denoted by $U \in \mathcal{S}(E, F)$, if U is not bounded below on any infinite-dimensional linear subspaces $M \subset E$. It is clear that $\mathcal{K}(E, F) \subset \mathcal{S}(E, F) \subset \mathcal{S}_p(E, F)$ for any Banach spaces E and F, and it is known that the classes $\mathcal{S}(E, F)$ and $\mathcal{S}_p(E, F)$ define norm-closed operator ideals in the sense of Pietsch [29] for any $1 \le p \le \infty$ (cf. [38, p. 289] for the case of $\mathcal{S}_p$).

Part of Theorem 1.1 is contained in the following dichotomy, which we also relate to the known characterisation of the compact composition operators on H^p.

Theorem 1.2. *Let* $1 \le p < \infty$ *and let* $\phi : \mathbb{D} \to \mathbb{D}$ *be any analytic map. Then either* $C_\phi \in \mathcal{K}(H^p)$, *or else* $C_\phi \notin \mathcal{S}_p(H^p)$. *Equivalently,* C_ϕ *fixes a copy of* ℓ^p *in* H^p *if and only if* (1.2) *does not hold.*

The above theorem holds for $p = 2$ because of the general fact due to Calkin that $\mathcal{K}(H^2) = \mathcal{S}(H^2) = \mathcal{S}_2(H^2)$ for the Hilbert space H^2, see e.g. [29, 5.1-5.2]. For $1 < p < \infty$ and $p \neq 2$ one has that

$$\mathcal{S}(H^p) = \mathcal{S}_2(H^p) \cap \mathcal{S}_p(H^p). \tag{1.4}$$

This follows from the characterisation of $\mathcal{S}(L^p)$ by Weis [38] combined with the well-known fact that $H^p \approx L^p \equiv L^p(0, 1)$, see e.g. [24, 2.c.17]. By contrast, for $p \neq 2$ all the inclusions

$$\mathcal{K}(H^p) \subsetneq \mathcal{S}(H^p), \quad \mathcal{S}(H^p) \subsetneq \mathcal{S}_2(H^p), \quad \mathcal{S}(H^p) \subsetneq \mathcal{S}_p(H^p) \tag{1.5}$$

are strict. This is easily deduced from the facts that $H^p \approx L^p$ contains complemented subspaces isomorphic to ℓ^p and ℓ^2, whereas any $U \in \mathcal{L}(\ell^p, \ell^q)$ is strictly singular for $p \neq q$, see e.g. [23, 2.c.3]. Thus Theorem 1.2 states that for $p \neq 2$ the compactness of composition operators $C_\phi \in \mathcal{L}(H^p)$ is a fairly rigid property as compared to (1.4) and (1.5) for arbitrary operators. It is also convenient to rephrase this as follows:

Corollary 1.3. *For $1 \le p < \infty$ the following conditions are equivalent for any analytic map $\phi : \mathbb{D} \to \mathbb{D}$:*

(i) *ϕ satisfies* (1.2),
(ii) *$C_\phi \in \mathcal{K}(H^p)$,*
(iii) *$C_\phi \in \mathcal{S}(H^p)$,*
(iv) *$C_\phi \in \mathcal{S}_p(H^p)$.*

The first result (excluding the case H^2) in the direction of Theorem 1.2 and Corollary 1.3 is due to Sarason [31], who showed that C_ϕ is weakly compact $H^1 \to H^1$ if and only if it is compact. Jarchow [15, p. 95] pointed out that as a consequence $C_\phi \in \mathcal{K}(H^1)$ if and only if C_ϕ is weakly conditionally compact on H^1, that is, $C_\phi \in \mathcal{S}_1(H^1)$ in view of Rosenthal's ℓ^1-theorem, see e.g. [23, 2.e.5]. Hence the case $p = 1$ in Theorem 1.2 and Corollary 1.3 was known earlier. We refer to Subsection 4.1 for a list of further references to analogous rigidity results for composition operators on several (classical) Banach spaces E of analytic functions on the unit disk $\mathbb{D}$.

The lattice structure of the operator norm-closed ideals of $\mathcal{L}(H^p) \approx \mathcal{L}(L^p)$ containing the compact operators is quite complicated for $1 < p < \infty$ and $p \neq 2$, see e.g. [29, 5.3.9] and [32]. For instance, $\mathcal{S}_p(H^p)$ and $\mathcal{S}_2(H^p)$ are mutually incomparable classes, since $H^p \approx L^p$ contains complemented copies of ℓ^2 and ℓ^p. However, note that Corollary 1.3 implies that if $C_\phi \in \mathcal{L}(H^p)$ fixes a copy of ℓ^2 in H^p, then C_ϕ must also fix a copy of ℓ^p in H^p. These facts raise the problem whether it is possible to explicitly determine the ℓ^2-singular composition operators on H^p. In turns out in Theorem 1.4 below that condition (1.3) characterises this class, thus providing a finer classification of the non-compact $C_\phi \in \mathcal{L}(H^p)$ for $1 \le p < \infty$ and $p \neq 2$. We stress that Theorem 1.4 (as well as the subsequent Theorem 1.5) does *not* hold for H^2.

Theorem 1.4. *Let $1 \le p < \infty$, $p \neq 2$, and $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic map. Then C_ϕ fixes a copy of ℓ^2 in H^p if and only if $m(E_\phi) > 0$. Equivalently, $C_\phi \in \mathcal{S}_2(H^p)$ if and only if* (1.3) *holds.*

Cima and Matheson [7] have shown that (1.3) characterises the completely continuous composition operators $C_\phi \in \mathcal{L}(H^1)$. As a significant strengthening of Theorem 1.4 we are further able to show that for $p > 1$ (and $p \neq 2$) condition (1.3) actually describes the operators C_ϕ which belong to the class $\mathcal{S}_{L^p}(H^p)$. Here $\mathcal{S}_{L^p}(H^p)$ is the maximal non-trivial ideal of $\mathcal{L}(H^p)$, see [10, p. 103]. To state the relevant result let h^p be the harmonic Hardy space consisting of the harmonic functions $f : \mathbb{D} \to \mathbb{C}$ normed by (1.1).

Theorem 1.5. *Let $1 < p < \infty$, $p \neq 2$, and $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic map. Then the following conditions are equivalent:*

(i) *ϕ satisfies $m(E_\phi) = 0$,*
(ii) *$C_\phi \in \mathcal{S}_{L^p}(H^p)$, that is, C_ϕ does not fix any copies of L^p in H^p,*
(iii) *$C_\phi \in \mathcal{S}_{L^p}(h^p)$,*
(iv) *$C_\phi \in \mathcal{S}_2(H^p)$.*

The paper is organised as follows. The proof of Theorem 1.2 is given in Section 2. The argument is based on explicit perturbation estimates, where the starting point is a known test function reformulation of the compactness criterion (1.2). The proofs of Theorems 1.4 and 1.5 are contained in Section 3. Although these results are connected we have stated them separately, since the argument for the ℓ^2-singularity in H^p also holds for $p = 1$. By contrast the proof of Theorem 1.5 relies on properties of $h^p = L^p(\mathbb{T}, m)$ for $1 < p < \infty$, and it depends on the non-trivial fact due to Dosev et al. [10] that the class $\mathcal{S}_{L^p}(L^p) \approx \mathcal{S}_{L^p}(h^p)$ is additive. Section 4 contains a number of further comments and open problems. As an application of Section 3 we characterise the ℓ^2-singular compositions $C_\phi \in \mathcal{L}(VMOA)$. As an additional motivation we also indicate a connection between a weaker version of Corollary 1.3 and a general extrapolation result [14] for operators on L^p-spaces.

A starting point for this paper was a question by Jonathan Partington about the strict singularity of composition operators on H^p for $p \neq 2$. We are indebted to Manuel González, Francisco Hernández and Dimitry Yakubovich for timely questions towards Theorems 1.4 and 1.5.

2. Proof of Theorem 1.2

For $a \in \mathbb{D}$ and fixed $0 < p < \infty$ let

$$g_a(z) = \frac{(1 - |a|^2)^{1/p}}{(1 - \overline{a}z)^{2/p}}, \quad z \in \mathbb{D}.$$

Here $\|g_a\|_p = 1$, since for $p = 2$ the corresponding function is the normalised reproducing kernel in H^2 associated to $a \in \mathbb{D}$. The proof of Theorem 1.2 is based on the following criterion: $C_\phi \in \mathcal{K}(H^p)$ *if and only if*

$$\limsup_{|a| \to 1} \|C_\phi(g_a)\|_p = 0. \tag{2.1}$$

This is a restatement using the test functions $(g_a) \subset H^p$ of a well-known characterisation of the compact operators $C_\phi \in \mathcal{L}(H^p)$ in terms of vanishing Carleson pull-back measures, see [9, Thm. 3.12.(2)] (such a characterisation was first obtained by MacCluer [27] in the case of $H^p(B_N)$ for $N > 1$, where B_N is the open euclidean ball in $\mathbb{C}^N$). Alternatively, (2.1) is stated explicitly for $p = 2$ in e.g. [33, 5.4], whereas the compactness of $C_\phi : H^p \to H^p$ is independent of $p \in (0, \infty)$ e.g. by [9, Thm. 3.12.(2)]. After these preparations we proceed to the proof itself.

Proof of Theorem 1.2. Suppose that $C_\phi \notin \mathcal{K}(H^p)$, where $1 \le p < \infty$. We will show by an explicit perturbation argument that C_ϕ fixes a linearly isomorphic copy of ℓ^p in H^p.

Since condition (2.1) fails there is $d > 0$ and a sequence $(a_n) \subset \mathbb{D}$ so that $|a_n| \to 1$ as $n \to \infty$ and

$$\|C_\phi(g_{a_n})\|_p \ge d > 0 \tag{2.2}$$

for all $n \in \mathbb{N}$. We may further assume without loss of generality that $a_n \to 1$ as $n \to \infty$. Namely, we may pass to a convergent subsequence in $\overline{\mathbb{D}}$ and compose ϕ with a suitable rotation of $\mathbb{D}$ that defines a linear isomorphism of H^p.

Our starting point is the phenomenon that (g_{a_n}) admits subsequences which are small perturbations of a disjointly supported sequence in $L^p(\mathbb{T}, m)$, and hence span an isomorphic copy of ℓ^p. The crux of the argument is that this can be achieved simultaneously for further subsequences of $(C_\phi(g_{a_n}))$, and the following claim actually contains the basic step of the argument:

Claim 2.1. There is a subsequence of (a_n), still denoted by (a_n) for simplicity, for which there are constants $c_1, c_2 > 0$ so that

$$c_1 \cdot \|(b_j)\|_{\ell^p} \le \Big\|\sum_{j=1}^{\infty} b_j C_\phi(g_{a_j})\Big\|_p \le c_2 \cdot \|(b_j)\|_{\ell^p} \quad \text{for all } (b_j) \in \ell^p. \tag{2.3}$$

Assuming Claim 2.1 momentarily, the proof of Theorem 1.2 is completed by using this claim a second time (formally in the case where $\phi(z) = z$ for $z \in \mathbb{D}$) to extract a further subsequence of (g_{a_n}), still denoted by (g_{a_n}), so that

$$d_1 \cdot \|(b_j)\|_{\ell^p} \le \Big\|\sum_{j=1}^{\infty} b_j g_{a_j}\Big\|_p \le d_2 \cdot \|(b_j)\|_{\ell^p} \quad \text{for all } (b_j) \in \ell^p, \tag{2.4}$$

for suitable constants $d_1, d_2 > 0$. Then by combining (2.3) and (2.4) we get

$$\begin{aligned} \Big\|\sum_{j=1}^{\infty} b_j C_\phi(g_{a_j})\Big\|_p &\ge c_1 \|(b_j)\|_p \\ &\ge c_1 {d_2}^{-1} \Big\|\sum_{j=1}^{\infty} b_j g_{a_j}\Big\|_p, \end{aligned}$$

so that the restriction of C_ϕ defines a linear isomorphism $M \to C_\phi(M)$, where $M = \overline{\text{span}}\{g_{a_j} : j \in \mathbb{N}\} \approx \ell^p$.

Let $A = \{\xi \in \mathbb{T} : \text{the radial limit } \phi(\xi) \text{ exists}\}$ and

$$E_\varepsilon = \{\xi \in A : |\phi(\xi) - 1| < \varepsilon\}$$

for $\varepsilon > 0$. Recall that $\mathbb{T} \setminus A$ has measure zero. The proof of Claim 2.1 is an argument of gliding hump type based on the following auxiliary observation.

Lemma 2.2. *Let ϕ and (g_{a_n}) be as above, where $a_n \to 1$ as $n \to \infty$. Then*

(L1) $\int_{\mathbb{T}\setminus E_\varepsilon} |C_\phi(g_{a_n})|^p dm \to 0$ *as* $n \to \infty$ *for each fixed* $\varepsilon > 0$,

(L2) $\int_{E_\varepsilon} |C_\phi(g_{a_n})|^p dm \to 0$ *as* $\varepsilon \to 0$ *for each fixed* $n \in \mathbb{N}$.

Proof. Observe first that

$$\int_{E_\varepsilon} |C_\phi(g)|^p dm \to 0$$

as $\varepsilon \to 0$ for any $g \in H^p$, since $\cap_{\varepsilon>0} E_\varepsilon = \{\xi \in A : \phi(\xi) = 1\}$ has measure 0 as ϕ is non-constant. Moreover, if $\varepsilon > 0$ is fixed and $\xi \in A \setminus E_\varepsilon$, then there is n_ε such that

$$|1 - \overline{a_n}\phi(\xi)| = |1 - \phi(\xi) + \phi(\xi)(1 - \overline{a_n})| \geq |1 - \phi(\xi)| - |1 - a_n| > \varepsilon/2$$

for all $n \geq n_\varepsilon$. It follows that

$$|C_\phi(g_{a_n})(\xi)|^p = \frac{1 - |a_n|^2}{|1 - \overline{a_n}\phi(\xi)|^2} \leq \frac{4(1 - |a_n|^2)}{\varepsilon^2},$$

so that (L1) holds as $n \to \infty$. □

To continue the argument of Claim 2.1 recall that $\int_{\mathbb{T}} |C_\phi(g_{a_n})|^p dm \geq d^p > 0$ by condition (2.2). We may then use Lemma 2.2 inductively to find indices $j_1 < j_2 < \dots$ and a decreasing sequence $\varepsilon_j > \varepsilon_{j+1} \to 0$ so that

(i) $\Big(\int_{E_{\varepsilon_n}} |C_\phi(g_{a_{j_k}})|^p dm\Big)^{1/p} < 4^{-n}\delta d$ for all $k = 1, \dots, n-1$,

(ii) $\Big(\int_{\mathbb{T}\setminus E_{\varepsilon_n}} |C_\phi(g_{a_{j_n}})|^p dm\Big)^{1/p} < 4^{-n}\delta d$,

(iii) $\Big(\int_{E_{\varepsilon_n}} |C_\phi(g_{a_{j_n}})|^p dm\Big)^{1/p} > d/2$

for all $n \in \mathbb{N}$. Here $\delta > 0$ is a small enough constant (to be chosen later). In fact, suppose that we have already found $a_{j_1}, \dots, a_{j_{n-1}}$ and $\varepsilon_1 > \dots > \varepsilon_{n-1}$ satisfying (i) - (iii). Then property (L2) from Lemma 2.2 yields $\varepsilon_n < \varepsilon_{n-1}$ such that

$$\Big(\int_{E_{\varepsilon_n}} |C_\phi(g_{a_{j_k}})|^p dm\Big)^{1/p} < 4^{-n}\delta d$$

for each $k = 1, \dots, n-1$. After this use property (L1) from Lemma 2.2 together with (2.2) to find an index $j_n > j_{n-1}$ so that conditions (ii) and (iii) are satisfied for the set E_{ε_n}.

In the interest of notational simplicity we relabel a_{j_n} as a_n for $n \in \mathbb{N}$. The idea of the argument is that the sequence $(C_\phi(g_{a_n}))$ essentially resemble disjointly supported peaks in $L^p(\mathbb{T}, m)$ close to the point 1. We will next verify the left-hand inequality in (2.3) by a direct perturbation argument. Let $b = (b_j) \in \ell^p$ be arbitrary. Our starting point will be the identity

$$\Big\|\sum_{j=1}^\infty b_j C_\phi(g_{a_j})\Big\|_p^p = \sum_{n=0}^\infty \int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} \Big|\sum_{j=1}^\infty b_j C_\phi(g_{a_j})\Big| p dm, \tag{2.5}$$

where we set $E_{\varepsilon_0} = \mathbb{T}$.

Observe first that for each $n \in \mathbb{N}$ we get that

$$\begin{aligned}\Big(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} |C_\phi(g_{a_n})|^p dm\Big)^{1/p} &= \Big(\int_{E_{\varepsilon_n}} |C_\phi(g_{a_n})|^p dm - \int_{E_{\varepsilon_{n+1}}} |C_\phi(g_{a_n})|^p dm\Big)^{1/p}\\ &> \Big((\frac{d}{2})^p - (4^{-n-1}\delta d)^p\Big)^{1/p} \ge \frac{d}{2} - 4^{-n-1}\delta d\end{aligned}$$

in view of (i) and (iii), where the last estimate holds because $0 < 1/p \le 1$. Moreover, note that

$$\Big(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} |C_\phi(g_{a_j})|^p dm\Big)^{1/p} < 2^{-n-j}\delta d$$

for all $j \neq n$. In fact, $\Big(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} |C_\phi(g_{a_j})|^p dm\Big)^{1/p}$ is dominated by $4^{-n}\delta d$ for $j < n$ and by $4^{-j}\delta d$ for $j > n$ in view of (i) and (ii).

Thus we get from the triangle inequality in L^p, together with the preceding estimates, that for all $n \in \mathbb{N}$ one has

$$\begin{aligned}&\Big(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} |\sum_{j=1}^\infty b_j C_\phi(g_{a_j})|^p dm\Big)^{1/p}\\ &\ge |b_n|\Big(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} |C_\phi(g_{a_n})|^p dm\Big)^{1/p} - \sum_{j\neq n} |b_j|\Big(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}} |C_\phi(g_{a_j})|^p dm\Big)^{1/p}\\ &\ge |b_n|(\frac{d}{2} - 4^{-n-1}d\delta) - 2^{-n}\delta d\|b\|_p \ge \frac{d}{2}|b_n| - 2^{-n+1}\delta d\|b\|_p.\end{aligned}$$

By summing over n we get from the disjointness and the triangle inequality in ℓ^p that

$$\begin{aligned}\Big\|\sum_{j=1}^\infty b_j C_\phi(g_{a_j})\Big\|_p &\ge \Big(\sum_{n=1}^\infty \Big| \frac{d}{2}|b_n| - 2^{-n+1}\delta d\|b\|_p \Big|^p\Big)^{1/p}\\ &\ge \frac{d}{2}\Big(\sum_{n=1}^\infty |b_n|^p\Big)^{1/p} - \delta d\|b\|_p\Big(\sum_{n=1}^\infty 2^{-(n-1)p}\Big)^{1/p}\\ &\ge d\Big(\frac{1}{2} - \delta\cdot(1-2^{-p})^{1/p}\Big)\|b\|_p \ge \frac{d}{4}\|b\|_p,\end{aligned}$$

where the last estimate holds once we choose $\delta > 0$ small enough, so that $\delta \cdot (1-2^{-p})^{1/p} < 1/4$.

The proof of the right-hand inequality in (2.3) is a straightforward variant of the preceding estimates. This inequality does not affect the choice of $\delta > 0$, and hence the details will be omitted here. $\square$

Remarks 2.3. The definitions of the classes $\mathcal{K}(H^p)$, $\mathcal{S}(H^p)$ and $\mathcal{S}_p(H^p)$ also make sense in the range $0 < p < 1$, where H^p are only quasi-Banach spaces. The composition operators C_ϕ are continuous on H^p for $0 < p < 1$, and

Theorem 1.2 as well as Corollary 1.3 remain true here. The argument is similar to the above, but the quasi-norms $\|\cdot\|_p$ in H^p as well as in ℓ^p are only p-norms, that is, $\|f+g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$ for $f, g \in H^p$. This will affect a few constants when applying the triangle inequalities as in the proof of Theorem 1.2. However, we leave the precise details for $0 < p < 1$ to the interested reader, since a full analogue of Theorem 1.1 appears out of reach. In fact, H^p and L^p are not isomorphic for $0 < p < 1$ (see e.g. [17, p. 35]), the structure of their respective subspaces differs (see e.g. [17, chapter 3.2]), and no version of (1.4) appears known.

Remarks 2.4. It is possible to ensure in the proof of Theorem 1.2 that the subspaces $M = \overline{\text{span}}\{g_{a_j} : j \in \mathbb{N}\}$ and $C_\phi(M)$ are both complemented in H^p (this also follows from the general result in [38] for $1 < p < \infty$). For this one uses the fact that the closed linear span of a disjointly supported sequence is complemented in $L^p(\mathbb{T}, m)$, a classical perturbation argument (cf. [23, 1.a.9]), as well as the complementation of $H^p \subset L^p(\mathbb{T}, m)$ for $1 < p < \infty$. Note also that for $p = 1$ the argument of Theorem 1.2 provides an alternative route to the weak compactness characterisation of Sarason [31] cited in Section 1. In fact, if $C_\phi \notin \mathcal{K}(H^1)$, then C_ϕ fixes a copy of the non-reflexive space ℓ^1 by Theorem 1.2, whence C_ϕ is not a weakly compact operator $H^1 \to H^1$.

3. Proof of Theorems 1.4 and 1.5

The proof of Theorem 1.4 is contained in the following three results. We first look separately at the case $p = 2$. Recall our notation $E_\phi = \{e^{i\theta} : |\phi(e^{i\theta})| = 1\}$ for analytic maps $\phi : \mathbb{D} \to \mathbb{D}$.

Lemma 3.1. *Suppose that condition* (1.3) *fails, that is,* $m(E_\phi) > 0$. *Then there exist integers* $0 \leq n_1 < n_2 < \cdots$ *and a constant* $K > 0$ *such that*

$$K^{-1} \cdot \|c\|_{\ell^2} \leq \left\| \sum_{k=1}^{\infty} c_k \phi^{n_k} \right\|_2 \leq K \cdot \|c\|_{\ell^2}$$

for all $c = (c_k) \in \ell^2$.

Proof. The upper estimate follows from the boundedness of C_ϕ on H^2 and the orthonormality of the sequence (z^n) in H^2.

To establish the lower estimate, note that $z^n \to 0$ weakly and therefore also $\phi^n = C_\phi(z^n) \to 0$ weakly in H^2 as $n \to \infty$. Hence we may set $n_1 = 0$ and then proceed inductively to pick increasing indices n_k such that the inner-products satisfy $|(\phi^{n_j}, \phi^{n_k})| \leq 2^{-2k} m(E_\phi)$ for all $1 \leq j < k$ and each $k \in \mathbb{N}$. Let $c = (c_k) \in \ell^2$ be arbitrary and note that

$$\left\| \sum_{k=1}^{\infty} c_k \phi^{n_k} \right\|_2^2 = \sum_{k=1}^{\infty} |c_k|^2 \|\phi^{n_k}\|_2^2 + 2 \operatorname{Re} \sum_{k=1}^{\infty} \sum_{j=1}^{k-1} c_j \bar{c_k} (\phi^{n_j}, \phi^{n_k}).$$

Obviously $\|\phi^{n_k}\|_2^2 \geq \int_{E_\phi} |\phi^{n_k}|^2 dm = m(E_\phi)$ for each k. Moreover, we get that

$$\Big|\sum_{k=1}^{\infty}\sum_{j=1}^{k-1} c_j \bar{c_k} (\phi^{n_j}, \phi^{n_k})\Big| \leq \|c\|_{\ell^2}^2 \sum_{k=1}^{\infty}\sum_{j=1}^{k-1} 2^{-2k} m(E_\phi)$$

$$\leq \tfrac{1}{2}\|c\|_{\ell^2}^2 m(E_\phi) \sum_{k=1}^{\infty}\sum_{j=1}^{k-1} 2^{-k}2^{-j} = \tfrac{1}{6}\|c\|_{\ell^2}^2 m(E_\phi).$$

By combining these estimates we obtain the desired lower bound

$$\Big\|\sum_{k=1}^{\infty} c_k \phi^{n_k}\Big\|_2^2 \geq \|c\|_{\ell^2}^2 m(E_\phi) - \tfrac{1}{2}\|c\|_{\ell^2}^2 m(E_\phi) = (\tfrac{1}{2}m(E_\phi))\|c\|_{\ell^2}^2.$$

□

In order to treat general $p \in [1, \infty)$ recall that the analytic map $f : \mathbb{D} \to \mathbb{C}$ belongs to *BMOA* if

$$|f|_* = \sup_{a\in\mathbb{D}} \|f \circ \sigma_a - f(a)\|_2 < \infty,$$

where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius-automorphism of $\mathbb{D}$ interchanging 0 and a for $a \in \mathbb{D}$. The Banach space *BMOA* is normed by $\|f\|_{BMOA} = |f(0)| + |f|_*$. Moreover, *VMOA* is the closed subspace of *BMOA*, where $f \in VMOA$ if

$$\lim_{|a|\to 1} \|f \circ \sigma_a - f(a)\|_2 = 0.$$

We refer to e.g. [12] and [13] for background on *BMOA*. It follows readily from Littlewood's subordination theorem that C_ϕ is bounded $BMOA \to BMOA$ for any analytic map $\phi : \mathbb{D} \to \mathbb{D}$, see e.g. [5, p. 2184].

The following proposition establishes one implication of Theorem 1.4.

Proposition 3.2. *Let $1 \leq p < \infty$ and suppose that $m(E_\phi) > 0$. Then there exist increasing integers $0 \leq n_1 < n_2 < \cdots$ such that the subspace*

$$M = \overline{\text{span}}\{z^{n_k} : k \geq 1\} \subset H^p$$

is isomorphic to ℓ^2 and the restriction $C_{\phi|M}$ is bounded below on M. Hence $C_\phi \notin \mathcal{S}_2(H^p)$.

Proof. We start by choosing the increasing integers (n_k) as in Lemma 3.1. By passing to a subsequence we may also assume that (z^{n_k}) is a lacunary sequence, that is, $\inf_k(n_{k+1}/n_k) > 1$. Paley's theorem (see e.g. [11, p. 104]) implies that for $1 \leq p < \infty$ the sequence (z^{n_k}) is equivalent in H^p to the unit vector basis basis of ℓ^2, that is,

$$\Big\|\sum_{k=1}^{\infty} c_k z^{n_k}\Big\|_p \sim \|c\|_{\ell^2} \qquad (3.1)$$

for all $c = (c_k) \in \ell^2$. (Here, and in the sequel, we use $\sim$ as a short-hand notation for the equivalence of the respective norms.)

Case $p \geq 2$. By Hölder's inequality and Lemma 3.1 we have that

$$\Big\| C_\phi\Big(\sum_{k=1}^\infty c_k z^{n_k}\Big)\Big\|_p = \Big\|\sum_{k=1}^\infty c_k \phi^{n_k}\Big\|_p \geq \Big\|\sum_{k=1}^\infty c_k \phi^{n_k}\Big\|_2 \sim \|c\|_{\ell^2}.$$

According to (3.1) and the boundedness of C_ϕ this proves the claim for $p \geq 2$.

Case $1 \leq p < 2$. We start by invoking a version of Paley's theorem for *BMOA* (see e.g. [13, Sec. 9]), which together with the boundedness of C_ϕ on *BMOA* ensures that

$$\Big\|\sum_{k=1}^\infty c_k \phi^{n_k}\Big\|_{BMOA} \leq \|C_\phi\| \cdot \Big\|\sum_{k=1}^\infty c_k z^{n_k}\Big\|_{BMOA} \leq K \cdot \|C_\phi\| \cdot \|c\|_{\ell^2}$$

for all $c = (c_k) \in \ell^2$ and a uniform constant $K > 0$. In view of Fefferman's H^1-*BMOA* duality pairing (see e.g. [13, Sec. 7]) we may further estimate

$$\begin{aligned}\Big\|\sum_{k=1}^\infty c_k \phi^{n_k}\Big\|_{BMOA}\Big\|\sum_{k=1}^\infty c_k \phi^{n_k}\Big\|_1 &\geq \Big|\big(\sum_{k=1}^\infty c_k \phi^{n_k}, \sum_{k=1}^\infty c_k \phi^{n_k}\big)\Big| \\ &= \Big\|\sum_{k=1}^\infty c_k \phi^{n_k}\Big\|_2^2 \sim \|c\|_{\ell^2}^2,\end{aligned}$$

where we again use Lemma 3.1 at the final step. By applying Hölder's inequality and combining the preceding estimates we obtain that

$$\Big\|\sum_{k=1}^\infty c_k \phi^{n_k}\Big\|_p \geq \Big\|\sum_{k=1}^\infty c_k \phi^{n_k}\Big\|_1 \geq K'\|c\|_{\ell^2}$$

for some uniform constant $K' > 0$. In particular, $C_\phi \notin \mathcal{S}_2(H^p)$ in view of (3.1), which completes the verification of the proposition for $1 \leq p < 2$. □

The converse implication in Theorem 1.4 is contained in the following

Proposition 3.3. *Let $1 \leq p < \infty$, $p \neq 2$, and suppose that $m(E_\phi) = 0$. If (f_n) is any normalized sequence in H^p which is equivalent to the unit vector basis of ℓ^2, then C_ϕ is not bounded below on $\overline{\text{span}}\{f_n : n \in \mathbb{N}\} \subset H^p$. In particular, $C_\phi \in \mathcal{S}_2(H^p)$.*

Proof. Assume to the contrary that

$$\Big\|\sum_{n=1}^\infty c_n C_\phi(f_n)\Big\|_p \sim \Big\|\sum_{n=1}^\infty c_n f_n\Big\|_p \sim \|c\|_{\ell^2}^2 \tag{3.2}$$

for all sequences $c = (c_n) \in \ell^2$. In particular, $\|C_\phi(f_n)\|_p \geq d > 0$ for all n and some constant d. We write $E_k = \{e^{i\theta} : |\phi(e^{i\theta})| \geq 1 - \frac{1}{k}\}$ for $k \geq 1$. Since

$\lim_{k\to\infty} m(E_k) = m(E_\phi) = 0$, we get that

$$\lim_{k\to\infty} \int_{E_k} |C_\phi(f_n)|^p \, dm = 0 \quad \text{for all } n.$$

On the other hand, $f_n \to 0$ weakly in H^p and hence $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. This implies that

$$\lim_{n\to\infty} \int_{\mathbb{T}\setminus E_k} |C_\phi(f_n)|^p \, dm = 0 \quad \text{for all } k.$$

By using the above properties and proceeding recursively in a similar fashion to the argument for Theorem 1.2 in section 2 we find increasing sequences of integers $0 \le n_1 < n_2 < \cdots$ and $1 = k_1 < k_2 < \cdots$, such that

$$\Big\| \sum_{j=1}^{\infty} c_j C_\phi(f_{n_j}) \Big\|_p^p = \sum_{l=1}^{\infty} \int_{E_{k_l}\setminus E_{k_{l+1}}} \Big| \sum_{j=1}^{\infty} c_j C_\phi(f_{n_j}) \Big|^p \, dm \sim \|c\|_{\ell^p}^p$$

holds for all $c = (c_j) \in \ell^p$ with uniform constants. However, for $p \neq 2$ such estimates obviously contradict (3.2). Thus $C_\phi \in S_2(H^p)$, and this completes the proof of the Proposition (and hence also of Theorem 1.4). □

We remind that Theorem 1.4 does not hold for $p = 2$. The result easily yields very explicit examples of operators $C_\phi \in \mathcal{S}_2(H^p) \setminus \mathcal{S}_p(H^p)$.

Example 3.4. Let $\phi(z) = \frac{1}{2}(1+z)$ for $z \in \mathbb{D}$. Theorem 1.4 implies that C_ϕ does not fix any copies of ℓ^2 in H^p. On the other hand, it is well known that $C_\phi \notin \mathcal{K}(H^p)$, see e.g. [34, Sec. 2.5], so that C_ϕ does fix copies of ℓ^p in H^p by Theorem 1.2.

We next prepare for the proof of Theorem 1.5. This involves the harmonic Hardy space h^p, that is, the space of complex-valued harmonic functions $f : \mathbb{D} \to \mathbb{C}$ normed by (1.1). Recall that for $1 < p < \infty$ there is a well-known isometric identification $h^p = L^p(\mathbb{T}, m)$ as a complex Banach space. Here $f \in h^p$ corresponds to its a.e. radial limit function $f \in L^p(\mathbb{T}, m)$, whereas conversely $g \in L^p(\mathbb{T}, m)$ determines the harmonic extension $P[g] \in h^p$ through the Poisson integral. Moreover, $h^p = H^p \oplus \overline{H_0^p}$, where $H_0^p = \{f \in H^p : f(0) = 0\}$ and $\overline{H_0^p} = \{\overline{f} : f \in H_0^p\}$.

Let $\phi : \mathbb{D} \to \mathbb{D}$ be any analytic map. The Littlewood subordination theorem for subharmonic functions (see e.g. [9, Thm. 2.22]) implies that the composition operator $f \mapsto f \circ \phi$ is also bounded $h^p \to h^p$ for $1 \le p < \infty$. It will be convenient in the argument to use the notation $\widetilde{C_\phi}(f) = f \circ \phi$ for $f \in h^p$ to distinguish the composition operator on h^p from its relative on H^p. In particular, if in addition $\phi(0) = 0$, then we may decompose

$$\widetilde{C_\phi} = \begin{pmatrix} C_\phi & 0 \\ 0 & C_\phi \end{pmatrix}, \quad \widetilde{C_\phi}(f, g) = (f \circ \phi, g \circ \phi), \tag{3.3}$$

as a matrix direct sum with respect to the decomposition $h^p = H^p \oplus \overline{H_0^p}$. Here $\phi(0) = 0$ ensures that $g \circ \phi \in H_0^p$ for any $g \in H_0^p$.

Proof of Theorem 1.5. We may assume during the proof that $\phi(0) = 0$. In fact, otherwise consider $\psi = \sigma_{\phi(0)} \circ \phi$, where $\sigma_{\phi(0)} : \mathbb{D} \to \mathbb{D}$ is the automorphism interchanging 0 and $\phi(0)$. Then $\psi(0) = 0$ and $\widetilde{C_\psi} = \widetilde{C_\phi} \circ \widetilde{C}_{\sigma_{\phi(0)}}$, where $\widetilde{C}_{\sigma_{\phi(0)}}$ is a linear isomorphism $h^p \to h^p$ (as well as $H^p \to H^p$), which does not affect any of the claims of the theorem.

The proof of the implication (iii) $\Rightarrow$ (i) is contained in the following claim.

Claim 3.5. Let $1 < p < \infty$ and suppose that $m(E_\phi) > 0$. Then $\widetilde{C_\phi} \notin \mathcal{S}_{L^p}(h^p)$, that is, there is a subspace $M \subset h^p$, $M \approx L^p$, such that $\widetilde{C_\phi}_{|M}$ is bounded below.

To prove the claim define the Borel measure ν on $\mathbb{T}$ by $\nu(A) = m((\phi)^{-1}(A))$. Then ν is absolutely continuous: if $A \subset \mathbb{T}$ is a Borel set and $u_A = P[\chi_A]$ is the harmonic extension (i.e. the Poisson integral) of χ_A, we have that

$$\nu(A) = \int_{(\phi^*)^{-1}(A)} dm \leq \int_{\mathbb{T}} u_A \circ \phi \, dm = u_A(\phi(0)) = u_A(0) = m(A).$$

Since $\nu(\mathbb{T}) = m(E_\phi) > 0$, it follows that the density $d\nu/dm \geq \delta$ for some $\delta > 0$ on a Borel set $F \subset \mathbb{T}$ of positive Lebesgue measure.

We may now choose $M = L^p(F, m)$. Indeed, given any $f \in L^p(F, m)$, we have

$$\|\widetilde{C_\phi} f\|_{L^p}^p \geq \int_{E_\phi} |f \circ \phi|^p \, dm = \int_{\mathbb{T}} |f|^p \, d\nu \geq \delta \int_F |f|^p \, dm = \delta \|f\|_{L^p(F,m)}^p,$$

which establishes Claim 3.5, since $L^p(F, m) \approx L^p$.

The implication (ii) $\Rightarrow$ (iii) follows from (3.3) and the non-trivial result that the class $\mathcal{S}_{L^p}(L^p) \approx \mathcal{S}_{L^p}(h^p)$ is additive, see [10, p. 103 and 105]. In fact, if $C_\phi \in \mathcal{S}_{L^p}(H^p)$, then

$$\widetilde{C_\phi} = \begin{pmatrix} C_\phi & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & C_\phi \end{pmatrix}$$

is the sum of two operators from $\mathcal{S}_{L^p}(h^p)$, and hence L^p-singular by additivity.

Finally, the proof of the implication (i) $\Rightarrow$ (ii) is already contained in that of Proposition 3.3. In fact, if there is a subspace $M \subset H^p$, $M \approx L^p$, so that C_ϕ is an isomorphism $M \to C_\phi(M)$, then C_ϕ also fixes the isomorphic copies of ℓ^2 contained in M. It was shown in Proposition 3.3 that the latter property is incompatible with (1.3). $\square$

We note that Claim 3.5 also holds for $p = 1$. However, there is no immediate analogue of Theorem 1.5 for H^1. In fact, $\mathcal{S}_{L^1}(H^1) = \mathcal{L}(H^1)$, since L^1 does not embed isomorphically into H^1, see e.g. [23, 1.d.1].

In conclusion, recall that there are infinitely many norm-closed ideals $\mathcal{I}$ of $\mathcal{L}(H^p)$ satisfying $\mathcal{S}_2(H^p) \subset \mathcal{I} \subset \mathcal{S}_{L^p}(H^p)$ for $1 < p < \infty$ and $p \neq 2$, see [29, 5.3.9]. By contrast, Theorems 1.4 and 1.5 imply that there is no corresponding gradation for composition operators on H^p. In some cases the trichotomy of Theorem 1.1 can be sharpened by combining with known results about the subspaces of $H^p \approx L^p$. For instance, for $2 < p < \infty$ it follows from a result of Johnson and Odell [16, Thm. 1] that if $C_{\phi|M}$ is bounded below on an infinite-dimensional subspace $M \subset H^p$ that contains no isomorphic copies of ℓ^2, then M embeds isomorphically into ℓ^p, whence $C_\phi \notin \mathcal{S}_p(H^p)$.

4. Concluding remarks and questions

In this section we list some further examples of Banach spaces of analytic functions, where composition operators have related rigidity properties, and draw attention to open problems. We also sketch another approach towards Theorem 1.2 which motivated this paper, though its conclusion is much weaker.

4.1. **Further rigidity properties.** The weaker rigidity property

$$C_\phi \in \mathcal{S}(E) \text{ if and only if } C_\phi \in \mathcal{K}(E) \tag{4.1}$$

holds for many other Banach spaces E of analytic functions on $\mathbb{D}$ apart from the Hardy spaces. The following list briefly recalls some cases. Typically these results were not stated in terms of strict singularity, and as a rule they do not yield as precise information as our results for H^p.

- The following dichotomy in [4, Thm. 1] is an explicit precursor of Theorem 1.2: *either* $C_\phi \in \mathcal{K}(H^\infty_v)$ *or* $C_\phi \notin \mathcal{S}_\infty(H^\infty_v)$. Here H^∞_v is the weighted H^∞-space for a strictly positive weight function v on $\mathbb{D}$. It is also possible to deduce versions of (4.1) for H^∞ (the case $v \equiv 1$) from even earlier results. In fact, it follows from any of the references [37], [3] or [8] that $C_\phi \in \mathcal{L}(H^\infty)$ is weakly compact if and only if $C_\phi \in \mathcal{K}(H^\infty)$. Moreover, Bourgain [6] established that $\mathcal{W}(H^\infty, X) = \mathcal{S}_\infty(H^\infty, X)$ for any Banach space X, where $\mathcal{W}$ denotes the class of weakly compact operators. Here $\mathcal{K}(H^\infty) \subsetneq \mathcal{S}(H^\infty)$, since this holds for the complemented subspace ℓ^∞ of H^∞.
- The dichotomy in Theorem 1.2 holds for arbitrary bounded operators on the Bergman space A^p. In fact, $A^p \approx \ell^p$ for $1 \le p < \infty$ by a result of Lindenstrauss and Pełcynski, see [39, Thm. III.A.11], whereas $\mathcal{S}(\ell^p) = \mathcal{S}_p(\ell^p) = \mathcal{K}(\ell^p)$ by a result of Gohberg, Markus and Feldman, see [29, 5.1-5.2].
- It is known that the Bloch space $\mathcal{B}$ is isomorphic to ℓ^∞, while $C_\phi \in \mathcal{W}(\mathcal{B})$ if and only if $C_\phi \in \mathcal{K}(\mathcal{B})$, see e.g. [25, Cor. 5]. Moreover, any $U \notin \mathcal{W}(\ell^\infty, X)$ fixes a copy of ℓ^∞ for any Banach space X, see [23, 2.f.4]. Consequently either $C_\phi \in \mathcal{K}(\mathcal{B})$ or $C_\phi \notin \mathcal{S}_\infty(\mathcal{B})$.
- It follows from [18, section 3] that $C_\phi \in \mathcal{K}(BMOA)$ if and only of $C_\phi \in \mathcal{S}_{c_0}(BMOA)$. In fact, the argument shows that if $C_\phi \notin \mathcal{K}(BMOA)$, then there is $M \subset VMOA$, $M \approx c_0$, so that $C_{\phi|M}$ is bounded below. Here again

$\mathcal{K}(BMOA) \subsetneq \mathcal{S}_{c_0}(BMOA)$, since $BMOA$ contains complemented subspaces isomorphic to ℓ^2 in view of Paley's theorem (see e.g. [13, Thm. 9.2]).

Actually the results of section 3 combined with [18] lead to a better understanding of the ℓ^2-singular composition operators on $VMOA$ and $BMOA$.

Proposition 4.1. *(i) If $\phi : \mathbb{D} \to \mathbb{D}$ is an analytic map, and $C_\phi \in \mathcal{S}_2(BMOA)$, then* (1.3) *holds (that is, $m(E_\phi) = 0$).*

(ii) If $\phi \in VMOA$, then $C_\phi \in \mathcal{S}_2(VMOA)$ if (and only if) (1.3) *holds.*

Proof. (i) The argument is essentially contained in that of Proposition 3.2. In fact, suppose that $m(E_\phi) > 0$, where $E_\phi = \{e^{i\theta} : |\phi(e^{i\theta})| = 1\}$. Then the proof of the case $1 \le p < 2$ of Proposition 3.2 gives a lacunary sequence (n_k) and constants $K_1, K_2 > 0$ so that in the H^1-$BMOA$ duality pairing

$$\Big\|\sum_{k=1}^{\infty} c_k \phi^{n_k}\Big\|_{BMOA} \Big\|\sum_{k=1}^{\infty} c_k \phi^{n_k}\Big\|_1 \ge K_1 \cdot \|c\|_{\ell^2}^2,$$

as well as $\| \sum_{k=1}^{\infty} c_k \phi^{n_k}\|_1 \le K_2 \cdot \|c\|_{\ell^2}$ for all $c = (c_k) \in \ell^2$. Since C_ϕ is bounded on $BMOA$ it follows as before from Paley's theorem in $BMOA$ that C_ϕ is bounded below on $\overline{\text{span}}\{z^{n_k} : k \in \mathbb{N}\} \approx \ell^2$ in $BMOA$.

(ii) Recall that $C_\phi : VMOA \to VMOA$ if $\phi \in VMOA$, see e.g. [5, Prop. 2.3]. Assume that $m(E_\phi) = 0$ and suppose to the contrary that there is a normalised sequence $(f_k) \subset VMOA$ equivalent to the unit vector basis of ℓ^2, for which

$$\Big\|\sum_{k=1}^{\infty} c_k C_\phi(f_k)\Big\|_{BMOA} \sim \|c\|_{\ell^2} \tag{4.2}$$

for all $c = (c_k) \in \ell^2$. In particular, $\|f_k \circ \phi\|_{BMOA} \ge d > 0$ for all k, while (f_k) is weak-null sequence in $VMOA$, so that $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Moreover, by the John-Nirenberg inequality there is a uniform constant $c > 0$ so that

$$\|f_k \circ \phi\|_4 \le c\|f_k \circ \phi\|_{BMOA}, \quad k \in \mathbb{N}.$$

Let $E_k = \{e^{i\theta} : |\phi(e^{i\theta})| \ge 1 - \frac{1}{k}\}$ for $k \in \mathbb{N}$. From the above estimates and Hölder's inequality we get that

$$\begin{aligned}\|f_n \circ \phi\|_2^2 &= \int_{E_k} |f_n \circ \phi|^2 dm + \int_{\mathbb{T}\setminus E_k} |f_n \circ \phi|^2 dm \\ &\le \Big(\int_{E_k} |f_n \circ \phi|^4 dm\Big)^{1/2} \sqrt{m(E_k)} + \int_{\mathbb{T}\setminus E_k} |f_n \circ \phi|^2 dm.\end{aligned}$$

Since $\int_{\mathbb{T}\setminus E_k} |f_n \circ \phi|^2 dm \to 0$ for each k as $n \to \infty$, we obtain that

$$\limsup_{n\to\infty} \|f_n \circ \phi\|_2^2 \le C\sqrt{m(E_k)}$$

for some constant $C > 0$ independent of $k \in \mathbb{N}$. By letting $k \to \infty$ and using that $m(E_\phi) = 0$ we deduce that $\lim_{n\to\infty} \|f_n \circ \phi\|_2 = 0$.

By [18, Prop. 6] there is a subsequence $(f_{n_k} \circ \phi)$ such that

$$\|\sum_{k=1}^{\infty} c_k f_{n_k} \circ \phi\|_{BMOA} \sim \|c\|_{\ell^\infty}$$

holds for all $c = (c_k) \in c_0$. Obviously this contradicts (4.2). $\square$

4.2. **An alternative approach.** We next indicate a different approach towards a weaker version of Theorem 1.2, which highlights a connection to the following general interpolation-extrapolation theorem for strictly singular operators on L^p-spaces due to Hernandez et.al. [14, Thm. 3.8]: *Let $1 \leq p < q \leq \infty$, and assume that the linear operator T is bounded $L^p \to L^p$ and $L^q \to L^q$. Moreover, suppose further that there is $r \in (p, q)$ for which $T \in \mathcal{S}(L^r)$. Then $T \in \mathcal{K}(L^s)$ for all $p < s < q$.*

To apply the above result suppose that $C_\phi \in \mathcal{S}(H^p)$, where $1 < p < \infty$. Recall from Section 3 that the related operator $f \mapsto \widetilde{C_\phi}(f) = f \circ \phi$ is bounded on the harmonic Hardy space h^p for $1 < p < \infty$, and that (3.3) holds with respect to $h^p = H^p \oplus \overline{H^p_0}$ provided $\phi(0) = 0$. It follows from (3.3) that $\widetilde{C_\phi} \in \mathcal{S}(h^p)$, since $\mathcal{S}(h^p)$ is a linear subspace. Fix q and r such that $1 < q < p < r < \infty$. Since $\widetilde{C_\phi}$ is bounded $h^t \to h^t$ for any $t \in (1, \infty)$ and $\widetilde{C_\phi} \in \mathcal{S}(h^p)$, the above extrapolation result applied to $h^t = L^t(\mathbb{T}, m)$ yields that $\widetilde{C_\phi} \in \mathcal{K}(h^s)$ for any $q < s < r$. In particular, $C_\phi \in \mathcal{K}(H^s)$ for any $q < s < r$ by restricting to $H^s \subset h^s$. Hence we have deduced by different means the following weak version of Theorem 1.2: *if $C_\phi \in \mathcal{S}(H^p)$, then $C_\phi \in \mathcal{K}(H^p)$ for $1 < p < \infty$.*

Above we do not address the technical issue that [14] only explicitly deals with real L^p-spaces, whereas the above application requires complex scalars. (We are indebted to Francisco Hernández for indicating that there is indeed also a complex version.) We leave the above alternative here as an incomplete digression, because it is not possible to obtain the full strength of Theorem 1.2 in this way (cf. the following example).

Example 4.2. We point out for completeness that the extrapolation result [14, Thm. 3.8] for $\mathcal{S}(L^p) = \mathcal{S}_p(L^p) \cap \mathcal{S}_2(L^p)$ does not have an analogue for the classes $\mathcal{S}_p(L^p)$ or $\mathcal{S}_2(L^p)$.

In fact, let (r_n) be the sequence of Rademacher functions on $[0, 1]$ and $f \mapsto Pf = \sum_{n=1}^{\infty} \langle f, r_n \rangle r_n$ the canonical projection $L^p \to M$ for $1 < p < \infty$, where $M = \overline{\text{span}}\{r_n : n \in \mathbb{N}\}$. Since $M \approx \ell^2$ by the Khinchine inequalities, see e.g. [23, 2.b.3], it follows that $P \in \mathcal{S}_p(L^p)$ by the total incomparability of ℓ^p and ℓ^2 for $p \neq 2$.

Furthermore, the results of section 3 (in particular, see Example 3.4 and (3.3)) imply that for $p \neq 2$ there are composition operators $\widetilde{C_\phi} \in \mathcal{S}_2(h^p)$ which fail to be compact.

4.3. **Open problems.** Our results suggest several natural questions.

Problems 4.3. (1) Are there results corresponding to our main theorems for $C_\phi \in \mathcal{L}(H^p, H^q)$ in the case $p \neq q$? Note that the conditions for boundedness and compactness of $C_\phi : H^p \to H^q$ are different in the respective cases $p < q$ and $q < p$, and they can be found in [30], [15] and [36]. For instance, for $q < p$ one has that $C_\phi \in \mathcal{K}(H^p, H^q)$ if and only if (1.3) holds. On the other hand, the class $\mathcal{S}(L^p, L^q)$ also behaves differently from (1.4) and (1.5) for $p \neq q$. For instance, if $p, q > 2$ and $p \neq q$, then $\mathcal{S}(L^p, L^q) = \mathcal{S}_2(L^p, L^q)$ but $\mathcal{S}_p(L^p, L^q) = \mathcal{L}(L^p, L^q)$. These equalities follow from the Kadec-Pełcynski dichotomy [1, 6.4.8] and the total incomparability of ℓ^p and ℓ^q.

(2) Is there an analogue of Theorem 1.5 for $p = 1$?

(3) Is the converse of Proposition 4.1.(i) also true?

(4) Is there a Banach space E of scalar-valued analytic functions on $\mathbb{D}$ and an analytic map $\phi : \mathbb{D} \to \mathbb{D}$, for which $C_\phi \in \mathcal{S}(E) \setminus \mathcal{K}(E)$? In this direction Lefevre et. al. [19] found a non-reflexive Hardy-Orlicz space H^ψ so that $C_\phi \in \mathcal{W}(H^\psi) \setminus \mathcal{K}(H^\psi)$, where ϕ is a lens map.

The approach sketched in Subsection 4.2 suggests that weaker rigidity properties such as (4.1) are likely to hold for many other concrete classes of operators on H^p. Subsequently Miihkinen [28] has used similar techniques as in section 2 to show that the dichotomy of Theorem 1.2 remains valid for the class of analytic Volterra operators T_g on H^p, where

$$f \mapsto (T_g(f))(z) = \int_0^z f(\tau) g'(\tau) d\tau, \quad z \in \mathbb{D}.$$

We refer e.g. to the surveys [2] or [35] for the conditions on the fixed analytic map $g : \mathbb{D} \to \mathbb{C}$ which characterise the boundedness or compactness of T_g.

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